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Global string in general relativity

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Abstract : In this paper, we have studied global strings retaining terms upto the order of $1/r^3$ in the energy-momentum tensor for a triplet scalar field, describing the string configuration. Also the gravitational field of the string solution has been considered. Finally, the geodesic of a test particle has been examined in the gravitational field of the string and the angle of deficit has been calculated for the string solution.

Keywords : Global string, general relativity, angle of deficit

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The cosmic string is one of the topological defects which may arise during phase transitions in the early universe [1-3]. These topological defects are characterised by the homotopy group of the vacuum manifold (μ) (In fact, $\pi_1(\mu) \neq I$ gives rise to cosmic strings). Strings have important astrophysical consequences, namely, the double quasar problem and galaxy formation can well be explained by strings [4]. The global string involves only a complex scalar field. For such a string, Harari and Sikirie [5] presented a solution of the linearlized field equations neglecting the radial variation of the scalar field outside the core of the string. Here, we retain the terms of order $1/r^3$ in the energy momentum tensor for a triplet scalar field and present an approximate solution of the field equations retaining terms upto order $1/r^3$. We also discuss the motion of the test particle by using the Hamilton-Jacobi method. Finally, we study the bending of light in the above field.

A Global string can be formed by global symmetry breaking in phase transition. The simplest case is that of a global $U(1)$ symmetry.

Consider the Lagrangian [6]

$$L = \nabla_\mu \phi^* \nabla^\mu \phi - \frac{1}{4}\lambda\left(\phi^*\phi - \eta^2\right)^2, \tag{1}$$

where ϕ is a complex scalar field.

The field configuration is

$$\phi = \eta f(r)\exp(i\theta). \tag{2}$$

The most general static cylindrically symmetric metric is

$$ds^2 = A(r)(-dt^2 + dz^2) + A(r)dr^2 + r^2B(r)d\theta^2, \tag{3}$$

here, the (t, z) plane is Lorentz invariant.

For the scalar field ϕ, we get the Lagrangian in terms of the real field f

$$L = \eta^2 \nabla_\mu \nabla_f^\mu + \eta^2 f^2 \nabla_\mu^\theta \nabla_\theta^\mu - \frac{1}{4}\lambda\eta^4(f^2 - 1)^2. \tag{4}$$

For the Lagrangian (4), we get the equation of motion

$$\nabla_\mu \nabla_f^\mu - f\nabla_\mu^\theta \nabla_\theta^\mu + \frac{1}{2}\lambda\eta^2 f(f^2 - 1) = 0. \tag{5}$$

For a flat space, we get the equation of motion

$$f'' + f'/r - f/r^2 - 1/2\,\delta^2 f(f^2 - 1) = 0 \tag{6}$$

(where a prime denotes differentiation *w.r.t.* r)

Here, $\delta = \left(\eta\sqrt{\lambda}\right)^{-1}$ is the core radius of the string.

We see that the approximate solution of (6) is [7]

$$f = 1 - \delta^2/r^2 \tag{7}$$

(upto $1/r^3$, *i.e.* neglecting $1/r^4$, $1/r^5$ *etc.*).

The energy momentum tensor characteristic of a static cylindrically symmetric cosmic string is

$$T_{\mu\gamma} = \frac{\partial L}{\partial g_{\mu\gamma}} - Lg_{\mu\gamma}. \tag{8}$$

Thus, we get

$$T_t^t = \frac{\eta^2 f'^2}{A} + \frac{\eta^2 f^2}{r^2 B} + \frac{1}{4}\lambda\eta^4(f^2 - 1)^2 = T_z^z, \tag{9}$$

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$$T_t^t = \frac{\eta^2 f'^2}{A} + \frac{\eta^2 f^2}{r^2 B} + \frac{1}{4}\lambda\eta^4 (f^2 - 1)^2, \tag{10}$$

$$T_\theta^\theta = \frac{\eta^2 f'^2}{A} + \frac{\eta^2 f^2}{r^2 B} + \frac{\lambda\eta}{4}(f^2 - 1)^2. \tag{11}$$

Putting the value of f in eqs. (9–11), we get

$$T_t^t = T_r^r = T_z^z = \frac{\eta^2}{r^2 B},$$

$$T_\theta^\theta = -\frac{\eta^2}{r^2 B} \tag{12}$$

(upto $1/r^3$, *i.e.* neglecting $1/r^4$, $1/r^5$ *etc.*).

The field equations of general relativity are consequently given by

$$\frac{A'}{A^2 r} + \frac{1}{2}\frac{A'B'}{A^2 B} + \frac{A'^2}{4A^3} = -\frac{8\pi G\eta^2}{r^2 B}, \tag{13}$$

$$\frac{A''}{A^2} - \frac{1}{2}\frac{A'^2}{A^3} = \frac{8\pi G\eta^2}{r^2 B}, \tag{14}$$

$$\frac{1}{2}\frac{A''}{A^2} + \frac{1}{2}\frac{B''}{AB} + \frac{B'}{rAB} - \frac{1}{2}\frac{A'^2}{A^3} - \frac{1}{4}\frac{B'^2}{AB^2} = -\frac{8\pi G\eta^2}{r^2 B}. \tag{15}$$

The solution of these equations are

$$A = 1 - 8\pi G\eta^2 \ln r,$$

$$B = 1 - 24\pi G\mu^2 \ln r \tag{16}$$

(neglecting $(\pi G\eta^2)^2$).

In this section, we shall study the motion of a test particle in the gravitational field of the global string described by

$$ds^2 = A(dt^2 - dz^2) - A dr^2 - r^2 B d\theta^2, \tag{17}$$

where $A = 1 - 8\pi G\eta^2 \ln r$; $B = 1 - 24\pi G\eta^2 \ln r$.

We consider a relativistic particle of mass m moving in the field of a global string (17).

The Hamilton Jacobi (H–J) equation is [8,9]

$$\frac{1}{A}\left[\left(\frac{\partial s}{\partial t}\right)^2 - \left(\frac{\partial s}{\partial z}\right)^2 - \left(\frac{\partial s}{\partial r}\right)^2\right] - \frac{1}{r^2 B}\left(\frac{\partial s}{\partial\theta}\right)^2 + m^2 = 0 \tag{18}$$

Now to determine the H–J function S from (18), we consider

$$S(t, z, r, \theta) = -Et + S_1(r) + Mz + J\theta, \tag{19}$$

where constants E, J can be identified as the energy and angular momentum of particle and M is the momentum along the z-direction.

If we substitute the ansatz (19) in (18), we get the expression of the unknown function $S_1(r)$ in the integral

$$S_1(r) = \varepsilon\int\left[E^2 - M^2 - \frac{A}{B}\frac{J^2}{r^2} + m^2 A\right]^{1/2} dr. \tag{20}$$

Here, $\varepsilon = \pm 1$ stands for the change of sign, whenever r passes through a zero of the integrand in (20).

In the H–J formalism, the path of the particle is characterized by $\frac{\partial S}{\partial E}$ = constant; $\frac{\partial S}{\partial M}$ = constant; $\frac{\partial S}{\partial J}$ = constant (Without loss of generality, we take the constants to be zero).

Thus, we get

$$t = \varepsilon\int E\left[E^2 - M^2 - \frac{A}{B}\frac{J^2}{r^2} + m^2 A\right]^{-1/2} dr, \tag{21}$$

$$z = \varepsilon\int M\left[E^2 - M^2 - \frac{A}{B}\frac{J^2}{r^2} + m^2 A\right]^{-1/2} dr, \tag{22}$$

$$\theta = \varepsilon\int\frac{JA}{Br^2}\left[E^2 - M^2 - \frac{A}{B}\frac{J^2}{r^2} + m^2 A\right]^{-1/2} dr. \tag{23}$$

From (21), we obtain the radial velocity of the particle as

$$\frac{dr}{dt} = \frac{1}{E}\left(E^2 - M^2 - \frac{A}{B}\frac{J^2}{r^2} + m^2 A\right)^{1/2} \tag{24}$$

The turning points of the trajectory are given by $dr/dt = 0$ as a consequence of the potential curve.

$$\frac{E}{m} = \left[(M^2/m^2 - 1)^2 + J^2/m^2 r^2 \right.$$

$$\left.(1 + 16\pi G\eta^2 \ln r) + 8\pi G\eta^2 \ln r\right]^{1/2}. \tag{25}$$

We see that the extremals of the potential curve are the solutions of the equation

$$\frac{J^2}{m^2}(1 - 8\pi G\eta^2) + 8\pi G\eta^2\frac{J^2}{m^2}\ln r^2 = 4\pi G\eta^2 r^2. \tag{26}$$

We see that it has two real solutions since $1 - 8\pi G\eta^2 > 0$. Hence, the trajectory of the particle can be trapped by the global string.

Thus, the global string has a gravitational field.

We shall now study the bending of light in the above field. The equation for the light track is [10]

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{l^2}{h^2}\frac{B^2}{A^2}r^4 - \frac{k^2}{h^2}\frac{B^2}{A^2}r^4 - r^2\frac{B}{A}. \tag{27}$$

The constants l, k, h are defined by

$$A\frac{dt}{d\xi} = l;\quad A\frac{dz}{d\xi} = k;\quad r^2 B\frac{d\theta}{d\xi} = h, \tag{28}$$

where ξ is an affine parameter along the light path.

Let $r = 1/U$

We get from (27)

$$\left(\frac{dU}{d\theta}\right)^2 = (a - U^2)\left[1 + \frac{b(2a - u^2)}{a - u^2}\ln u\right], \tag{29}$$

where $a^2 = \frac{l^2 - k^2}{h^2}$, $b = 16\pi\eta^2 G$

[neglecting $(\pi G\eta^2)^2$]

From (29), we get

$$\theta = \sin^{-1} z + \cos^{-1} zb \ln a - bz(\ln z - 1), \qquad (31)$$

where $z = U/a$

[neglecting u^4 and $u^2 \pi G \eta^2$].

Now $U = 0$ corresponds to

$$\theta = \pi/2 b \ln a,$$

hence, the bending of light has the expression

$$\pi(1 - b \ln a).$$

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